Grade 6 Math Circles

February 20-22, 2024

Arithmetic Sequences - Problem Set

1. Determine which sequences are arithmetic, geometric, both, or neither.

   (a) $1,\ 5,\ 9,\ 13,\ 17,\ 21$

   (b) $9,\ 7,\ 5,\ 4,\ 2,\ 0$

   (c) $64,\ 32,\ 16,\ 8,\ 4\ ,2,\ 1$

   (d) $-10,\ 0,\ 10,\ 20,\ 30,\ 40$

   (e) $3,\ 3,\ 3,\ 3\ ...$

   Solution:

   (a) This is an arithmetic sequence, since there is a common difference of 4 for every term in the sequence.

   (b) This sequence is neither arithmetic nor geometric. The common difference for the first three terms is 2, then 1 for the third to fourth term, then 2 again for the last three terms. Since the common difference changes, the sequence is not arithmetic. It is also not geometric since we cannot multiply each term by the same number each time to get the next.

   (c) We notice that to get to the next term in the sequence, we have to divide the previous number by 2. This repeated pattern tells us it is a geometric sequence.

   (d) Each term has a common difference of 10, therefore this is an arithmetic sequence.

   (e) When we have any repeated number for a sequence, there are two ways we can think about it:

      (i) We are adding 0 to the previous term to give us the next term

      (ii) We are multiplying the previous term by 1 to give us the next term.

      Case (i) corresponds to an arithmetic sequence, while case (ii) corresponds to a geometric sequence. Therefore, this sequence is both arithmetic *and* geometric.

2. An arithmetic sequence has a common difference of 5. If the first term is 10 and the last term is 165, how many terms are in the sequence? How does this change if the first term is -10?

Solution:
We're told the common difference is 5, therefore $d = 5$. Also, we know the first term is 10 so $a = 10$. Finally, we're looking for the number of terms n it takes us to get to the final term $b = 165$. We use the formula $b = a + (n-1)d$ to solve for n:

$$b = a + (n-1)d$$

$$165 = 10 + (n-1) \cdot 5$$

$$165 - 10 = \cancel{10} - \cancel{10} + (n-1) \cdot 5$$

$$155 = (n-1) \cdot 5$$

$$\frac{155}{5} = \frac{(n-1) \cdot \cancel{5}}{\cancel{5}}$$

$$31 = n - 1$$

$$\therefore n = 32$$

So 32 terms are needed to reach the value of 165.
If instead our sequence started at -10, we could plug $a = -10$ into the same formula, or we could recognise that it takes 4 jumps of our common difference of 5 to get to our original starting point of 10, so it will take us an additional 4 terms to get from -10 to 165. In other words, we know our common difference is 5, so if we start our new sequence at $a = -10$, we must jump four times to reach our original starting point of 10:

$$-10 \to -5 \to 0 \to 5 \to 10$$

We found $n = 32$ when $a = 10$, which means $n = 32 + 4 = 36$ when $a = -10$. This is much quicker than reusing the formula!

3. Consider the arithmetic sequence

$$5,\ 11,\ 17,\ 23,\ 29$$

(a) Determine the values of a, b, n and d.

(b) Use these values to find the sum of the sequence. Verify your answer with a calculator.

Solution:

(a) We see our first and last terms of the sequence are $a = 5$ and $b = 29$. Next, our common difference is $d = 11 - 5 = 6$. There are also a total of five terms, so $n = 5$.

(b) The sum of an arithmetic sequence is given by the formulae:

$$S = n \cdot \frac{a+b}{2} = \frac{n}{2} \cdot (2a + d \cdot (n-1))$$

The sum formula we should pick depends on the information we know. From part (a), we know $a = 5$, $b = 29$, $d = 6$ and $n = 5$. This means we can use either formula. For simplicity, let's use the formula $S = n \cdot \frac{a+b}{2}$. Plugging in our values, we get

$$S = n \cdot \frac{a+b}{2}$$
$$S = 5 \cdot \frac{5+29}{2}$$
$$S = 5 \cdot \frac{34}{2}$$
$$S = 5 \cdot 17$$
$$S = 85$$

We can verify this in our calculator by just adding them manually. We see that

$$5 + 11 + 17 + 23 + 29 = 85$$

as expected.

4. To try and reduce her use of oil-based fuel, Maura purchases an electric vehicle that takes 2,700 gallons of diesel fuel to manufacture, but requires no gasoline to use once its made. Meanwhile, the average gas-powered car only takes about 260 gallons of diesel fuel to manufacture, but uses one gallon of gasoline per 22 miles travelled. Create an arithmetic sequence to model the fuel consumption of both types of vehicles, then determine how many miles must be travelled for both vehicles to consume equal amounts of fuel in their lifetimes. What happens after this point?

Solution: Although the electric vehicle takes much more gasoline-energy to produce the car, it requires *none* once it's being used. The arithmetic sequence for electric vehicle's fuel consumption is simply

$$2,700,\ 2,700,\ 2,700,\ \ldots$$

. However, every time someone travels 22 miles in the gas-powered vehicle, another gallon of gasoline is needed to keep it running. The gas-powered vehicle's fuel consumption can be modeled by the arithmetic sequence

$$260,\ 261,\ 262,\ \ldots$$

where each consecutive term represents another 22 miles traveled. To find the number of terms this sequence needs to get to 2,700 gallons, we use the Arithmetic Sequences Formula where $a = 260$, $b = 2,700$ and $d = 1$, since the difference between each consecutive term is 1 gallon. We want to find n.

$$b = a + d \cdot (n - 1)$$

$$2,700 - 260 = \cancel{260} + 1 \cdot (n - 1) - \cancel{260}$$

$$2,440 = n - 1$$

$$\therefore n = 2,441$$

So there are a total of 2,441 terms in the sequence, which means the gas-powered vehicle must drive 22 miles a total of 2,441 times to use the same amount of fuel as the electric vehicle. The total number of miles traveled is then $22 \cdot 2,441 = 53,702$ miles. Therefore, after 53,702 miles traveled, the total amount of gas consumed by the gas-powered vehicle's creation and usage will surpass the amount of gas used to produce the electric car. Initially it may seem like electric cars actually use more fuel, but after a long time of using each, the electric car eventually uses less total fuel throughout its lifetime than the gas-powered car.

5. Find the sum of the sequence below:

$$-12,\ -31,\ -50,\ \ldots\ -1190,\ -1209,\ -1228$$

6. What is the sum of the first 30 negative odd numbers?

Solution: The sequence of the first odd numbers starts as

$$-1, \ -3, \ -5, \ -7, \ \dots$$

This sequence continues for 60 terms total, therefore $n = 60$. The sequence starts at $a = -1$, and since each term decreases by 2 each jump to the right, the common difference is $d = -2$. This is all the information we need to determine the sum of the sequence. We use the formula

$$\begin{aligned} S &= \frac{n}{2}(2a + d \cdot (n-1)) \\ &= \frac{60}{2}(2(-1) - 2 \cdot (60-1)) \\ &= 30(-2 - 2 \cdot 59) \\ &= 30 \cdot -120 \\ &= 3,600 \end{aligned}$$

7. Alicia has 20 shares of stock in the Euclid corporation. Each share has an initial value of \$10. The share value increases each week according to the arithmetic sequence: \$10, \$13, \$16, ...

   (a) If Alicia waits 20 weeks and then sells all of her shares, how much money will she have?

   (b) If Alicia sells one share at \$10 and then one share per week thereafter until she has no shares left, how much money will she have?

Solution:

(a) Since Alicia sells all of her shares at once, they will all have the same value. The value of each share after 20 weeks will be the value of the 20^{th} term in the sequence \$10, \$13, \$16, ... To determine this value, we use the Arithmetic Sequences Formula and set $a = \$10$, $n = 20$ and $d = \$13 - \$10 = \$3$ to solve for b.

$$\begin{aligned} b &= a + d \cdot (n-1) \\ &= \$10 + \$3 \cdot (20-1) \\ &= \$10 + \$3 \cdot (19) \end{aligned}$$

$$= \$10 + \$57$$

$$\therefore b = \$67$$

Finally, since she has 20 total shares, she will have $\$67 \cdot 20 = \$1,340$ after 20 weeks.

(b) If Alicia sells one share per week, the share value will correspond to each term in the original sequence. Each week, the value of the share will increase by \$3. To find the total amount of money she has after 20 weeks, we must find the sum of the first 20 terms of the sequence $\$10,\ \$13,\ \$16,\ ...$ After setting $a = \$10$, $n = 20$ and $d = \$3$, we use our sum formula:

$$S = \frac{n}{2}(2a + d \cdot (n-1))$$

$$S = \frac{20}{2}(2(\$10) + \$3 \cdot (20-1))$$

$$S = 10 \cdot (\$20 + \$57)$$

$$S = 10(\$77)$$

$$S = \$770$$

So after 20 weeks of selling one share per week, Alicia will have \$770 total.

8. The first 45 terms of an arithmetic sequence are:

$$-308,\ -301,\ -294\ ...,\ -21,\ -14,\ -7$$

The sum of the **entire sequence** is zero. Without using any formulas, explain how we could predict the total number of terms in the sequence.

Solution: We notice that because the common difference of this sequence is 7, its next term would be 0, which contributes nothing to the sum. The next number would be 7, then 14, then 21, and so on. It is *symmetric* to the first 45 terms, which means it has the same terms on the right side of zero, except now they are positive. If the total sum is zero, every positive number must exactly cancel out all 45 negative numbers. Therefore, there must be 45 positive numbers in the sequence. So the total number of terms in the sequence is 45 negative numbers plus 45 positive numbers plus the one middle term for 0.

$$\implies n = 45 + 45 + 1 = 91$$

9. Create a sequence of nine consecutive even numbers whose sum is exactly 900.

Solution: The word "consecutive" means "in a row". Consecutive even numbers like 2, 4, 6, 8 ... have a difference of 2, so a sequence of consecutive even numbers has a common difference of $d = 2$. We are trying to create a sequence of 9 numbers, which tells us that $n = 19$. Finally, the sum of all of the terms in the sequence is $S = 900$. We are now ready to use the formula

$$S = \frac{n}{2}(2a + d \cdot (n - 1))$$

$$900 = \frac{9}{2}(2a + 2 \cdot (9 - 1))$$

$$900 = \frac{9}{2}(2a + 2 \cdot 8)$$

$$\frac{900}{\frac{9}{2}} = \frac{\cancel{\frac{9}{2}}(2a + 16)}{\cancel{\frac{9}{2}}}$$

$$200 - 16 = 2a + \cancel{16} - \cancel{16}$$

$$\frac{184}{2} = \frac{\cancel{2}a}{\cancel{2}}$$

$$a = 92$$

Now that we know our first term is 92, our sequence of consecutive even numbers is

$$92,\ 94,\ 96,\ 98,\ 100,\ 102,\ 104,\ 106,\ 108$$

We can also easily verify that the sum of each number in this sequence is

$$92 + 94 + 96 + 98 + 100 + 102 + 104 + 106 + 108 = 900$$

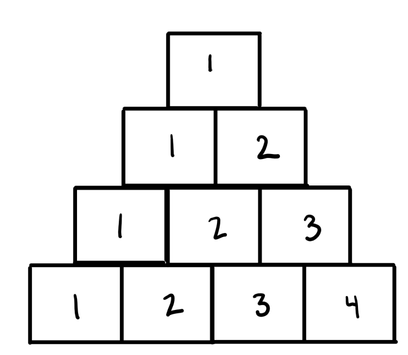
If we wanted to solve this problem even quicker, we could notice that 9 terms fitting into a total sum of 900 would give us an average value of $\frac{900}{9} = 100$ per term. Since we have

9 consecutive even terms, we would choose our middle term to be our average (100) and make 4 consecutive jumps of $+2$ to the right and 4 jumps of -2 to the left of the middle term, 100. Making 4 jumps on either side of the middle term gives us $4+4=8$ terms, plus the middle term which gives 9 terms total. Our sequence would remain the same as before:

$$92,\ 94,\ 96,\ 98,\ 100,\ 102,\ 104,\ 106,\ 108$$

This method of splitting an arithmetic sequence into two "halves" is precisely how the formula $S = n \cdot \left(\frac{a+b}{2}\right)$ works, as we briefly saw in the lesson. In this case, $n = 9$ and the $\frac{a+b}{2} = \frac{92+108}{2} = 100$ so $S = 9 \cdot 100 = 900$, as we wanted.

10. A pyramid of rectangular blocks is drawn below. Each row of the pyramid corresponds to the number of blocks in that row. For example, the first row has one block, the second row has two blocks, and so on.



(a) How many blocks are there in total for a block pyramid with 500 rows?

(b) If each block is 2.5 pounds, how much does this row weigh?

Solution:

(a) Since every row's number of blocks increases by one each time, this pyramid of blocks can be represented by an arithmetic sequence with common difference $d = 1$, starting at $a = 1$:

$$1,\ 2,\ 3,\ 4,\ \ldots$$

We use the sum formula

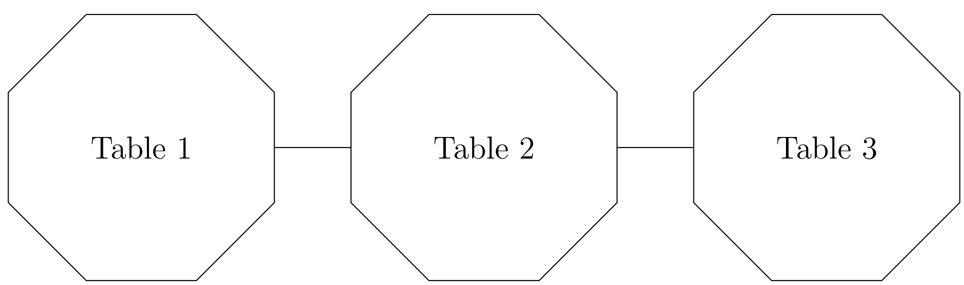
$$S = \frac{n}{2}(2a + d \cdot (n-1))$$

where n represents the number of rows and S represents the total sum of blocks. We then know $n = 500$. Plugging this into our formula, we get:

$$S = \frac{500}{2}(2(1) + (1) \cdot (500 - 1))$$
$$= 250 \cdot (2 + 499)$$
$$= 250 \cdot 501 = 125,250$$

(b) We're told that the row number for this block pyramid corresponds to how many blocks there are in that row. This means that the 500^{th} row will have 500 blocks. If each block weighs 2.5 pounds, then the total weight of the 500^{th} row is $500 \cdot 2.5 = 1,250$ pounds.

11. * A restaurant has tables in the shape of an octagon which hold one person per edge of the octagonal table. However, when the tables are joined for larger parties, the sides joining another table can no longer be used for seating. If the restaurant were big enough to hold large crowds of people, use sequences to find how many tables are needed to seat exactly 200 people.



Solution: We can model the total number of people seated by using an arithmetic sequence where each term represents how many people are seated for n number of tables. First, we must realise that the outer two tables each seat 7 people, since only one side of the octagon connects to another table. These two tables of 7 people seat a total of 14 people. Now we just need to seat $200 - 14 = 186$ more people to reach our total of 200 people seated.
Since 14 people are already seated, we can create a new sequence which increases by 6 every time n increases by 1. This is because for every table we add, we are able to seat 6

more people. Remember, though, that each of the remaining tables seat 6 people, so when the sequence starts at $n = 1$, we must start the sequence at $a = 14 + 6 = 20$. If we were to start the sequence at $a = 14$, we would be ignoring the 6 people that the first table ($n = 1$) would hold. Our arithmetic sequence representing the number of people seated is

$$20,\ 26,\ 32,\ \ldots\ 200$$

We have our first term $a = 20$, last term $b = 200$, and common difference $d = 6$. To find n, we put these values into our arithmetic sequence formula:

$$b = a + d \cdot (n - 1)$$

$$200 = 20 + 6 \cdot (n - 1)$$

$$200 - 20 = \cancel{20} - \cancel{20} + 6 \cdot (n - 1)$$

$$180 = 6 \cdot (n - 1)$$

$$\frac{180}{6} = \frac{\cancel{6} \cdot (n - 1)}{\cancel{6}}$$

$$30 = n - 1 \implies n = 31$$

So there will be 31 inner tables.

Finally, we mustn't forget the two outer tables. There are 31 inner tables and 2 outer tables, so 33 total tables are needed to seat 200 people.